Existence and non existence of a ground state for the massless Nelson model under binding condition

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Abstract

We consider a model describing N non-relativistic particles coupled to a massless quantum scalar field, called *Nelson model*, under a binding condition on the external potential. We prove that this model does not admit ground state in the Fock representation of the canonical commutation relations, but it does in another not unitarily equivalent coherent representation. Remark that the binding condition is satisfied for small values of the coupling constant.

Keywords: ground state, infrared problem, Nelson model.

1 Introduction

When considering a non-relativistic atom coupled to a quantized radiation field, it is natural to require that the model predicts the existence of a ground state. If the field is massive, this usually follows from the fact that the bottom of the spectrum is an isolated point. On the other hand, in the massless case the bottom of the spectrum lies in the continuum.

For the standard model of non-relativistic Quantum Electrodynamics (often called Pauli-Fierz model) with N-body Coulomb interactions, the existence of a ground state in the massless case was first established by Bach, Fröhlich and Sigal in [4] for sufficiently small values of some parameters in the theory. Subsequently, Griesemer, Lieb and Loss proved in [13] and [14] that a ground state exists for all values of the parameters under the following *binding condition*. Let us call E_N the bottom of the spectrum of the N-particle Hamiltonian with external potential V and E_N^0 its translation invariant part (i.e. V is removed). Then the *binding condition* is

$$E_N < E^0_{N-N'} + E_{N'} \quad \text{for all} \quad N' < N. \qquad (B)$$

If the field is neglected i.e. in the framework of usual Schrödinger operators, this condition is equivalent to $E_N < E_{N-1}$ since $E_N^0 = NE_1^0 = 0$, and it is satisfied for N-body Coulomb interactions if $N < Z+1$, where Z the charge of the nucleus, as proved long ago by Zhislin in [22]. In [5], Barbaroux, Chen and Vulgater showed that *(B)* is also satisfied for the standard model of non- relativistic QED for $N = 2$, $N < Z+1$. Finally, in [17] Lieb and Loss completed the picture by proving the statement for any N provided $N < Z+1$.

A natural question to ask is whether the infrared behaviour of other (simplified) non-relativistic QED models is the same.

In this paper we consider a model describing N scalar non-relativistic particles (fermions) coupled to a scalar Bose field. This is usually called the *Nelson model*. The Hamiltonian for N particles is given by

$$H_N = K_N \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega) + \lambda\Phi(v). \qquad (1.1)$$

Here K_N is a Schrödinger operator describing the dynamics of the particles, λ is a coupling constant,

$$\mathrm{d}\Gamma(\omega) := \int |k| a^*(k) a(k) \mathrm{d}k,$$

and

$$\Phi(v) := \frac{1}{\sqrt{2}} \sum_{j=1}^{N} \int \frac{e^{-\mathrm{i}k\mathrm{x}_j}}{|k|^{1/2}} \rho(k) a^*(k) \mathrm{d}k + h.c,$$

where $a^*(k), a(k)$ are the usual creation and annihilation operators, ρ an ultraviolet cutoff function. These objects will be described more precisely in the next section.

The Nelson model belongs to a class of Hamiltonian, often called *abstract Pauli-Fierz hamiltonians*, which includes the so called *generalized spin-boson models*, and for which the problem of the ground state has been studied in recent years (see for instance [2],[3], [12],[11] and references therein). In the case of a *confining* external potential, it is known that the Nelson model does not admit a ground state in the Fock representation of the canonical commutation relations (CCR) due to, heuristically speaking, too many soft photons ([12], [18]). Nevertheless, it is possible to find another representation of the CCR where the ground state exists as done by Arai in [1]. This representation is not unitarily equivalent to the Fock one ([1]). This is called *infrared catastrophe*. The infrared problem also appears in scattering theory. This was first studied by Fröhlich in [10], and more recently by Pizzo in [20] and Chen [6].
Here we consider N-body interactions, more precisely we take:

$$K_N = \sum_{i=1}^{N} -\frac{1}{2} \triangle_{\mathrm{x}_i} + V(X) + I(X) \tag{1.2}$$

where V, I satisfy the following:

$$\left.\begin{array}{ll}
(i) & V(X) = \sum_{j=1}^{N} v(\mathrm{x}_j) \quad I(X) = \sum_{i<j}^{N} u(\mathrm{x}_i - \mathrm{x}_j) \qquad v(\mathrm{x}), u(\mathrm{x}) : \mathbb{R}^3 \to \mathbb{R}, \\
(ii) & v = v_{sing} + v_{reg}, u = u_{sing} + u_{reg},\ \textit{where } v_{sing} \textit{ and } u_{sing} \textit{ have} \\
 & \textit{compact support},\quad v_{reg} \textit{ and } u_{reg} \textit{ are continuous}, \\
(iii) & v_{reg}(\mathrm{x}) = O(|\mathrm{x}|^{-\varepsilon_1}) \quad \textit{for } |\mathrm{x}| \to \infty \qquad \textit{where} \quad \varepsilon_1 > 0, \\
 & u_{reg}(\mathrm{x}) = O(|\mathrm{x}|^{-\varepsilon_2}) \quad \textit{for } |\mathrm{x}| \to \infty \qquad \textit{where} \quad \varepsilon_2 > 0, \\
(iv) & v \textit{ and } u \textit{ are } -\triangle \textit{ bounded with relative bound zero.}
\end{array}\right\} \textit{(I)}$$

We will also consider the Nelson model in another representation of the CCR, the same used in [1], obtaining a new Hamiltonian denoted by H_N^{ren}, described in the subsection 2.2.6.

We prove the following:

Theorem 1.1. *Assume the binding condition (B) and the hypothesis (I) on the potentials. Then H_N has no ground state.*

Theorem 1.2. *Assume the binding condition (B) and the hypothesis (I) on the potentials. Then H_N^{ren} admits a ground state.*

Moreover, if $v = -\frac{Z}{|\mathrm{x}|}$ and $u(\mathrm{x}) = \frac{1}{|\mathrm{x}|}$, (B) is clearly satisfied for λ small enough as explained in Proposition 3.1.

A key point in the proofs is to guarantee that any candidate to be a ground state must be localized in the fermion variables. In the confined case this property follows easily from the compactness of $(K_N + \mathrm{i})^{-1}$, while in our case it requires some work. In particular, if $N > 1$, one has to deal with photon localization which greatly complicates the proofs. This problem however, was already solved for the more involved standard

model in [13] and [14], and the same proof does apply to our case. Here we only take care to write down explicitly that the estimates we obtain are uniform in the infrared cutoff parameter μ, which was, since the proof was split over two papers, somewhat left to the reader.
Once the localization property is guaranteed, we can adapt techniques in [12] to both existence and non-existence of the ground state. Once again the proofs in [12] make large use of the compactness of $(K_N + \mathrm{i})^{-1}$, which does not hold anymore, but we can circumvent this difficulty using fermion localization in a more direct way.

In the one-particle case, the same problem has already been approached in [16] by Hirokawa (non-existence in Fock representation) and by Sasaki in [21] (existence in another representation).

The paper is organized as follows. In section 2 we introduce precisely the objects. In section 3 we make some useful remarks about the binding condition. Section 4 is devoted to exponential decay. Finally sections 5 and 6 are devoted respectively to the proofs of Theorem 1.1 and Theorem 1.2.

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2 Definition and basic constructions

2.1 Notation

We shall use the following notation:

Definition. Let f be a function in $C_0^\infty(\mathbb{R}^d)$, we denote by f_R the operator of multiplication by $f(\frac{\mathrm{x}}{R})$ in $L^2(\mathbb{R}^d, \mathrm{d}x)$.

Definition. Let $A : \mathbb{R} \ni R \mapsto A_R$ where A_R is a self-adjoint operator on a Hilbert space $\mathcal{H}$ and let B be a self-adjoint operator, $B \geq 0$. We say that $A_R = O(R^n)B$ if for $R >> 1$ $\mathcal{D}(|A_R|^{1/2}) \supset \mathcal{D}(B^{1/2})$ and $\pm A_R \leq C(R)B$ where $C(R) = O(R^n)$. We say that $A_R = O(R^n)$ if $A_R = O(R^n)\mathbb{1}$.

If A, B are two operators on a Hilbert space, we set $\mathrm{ad}_A B := [A, B]$. The precise meaning of $[A, B]$ will be either specified or clear from the context.

Definition. Let A be an operator on a Hilbert space $\mathcal{H}_1$ and B an operator on a Hilbert space $\mathcal{H}_2 \otimes \mathcal{H}_1$. We introduce $T : \mathcal{H}_2 \otimes \mathcal{H}_1 \otimes \mathcal{H}_1 \to \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_1$, $T(\psi \otimes u \otimes v) := u \otimes \psi \otimes v$, and we define *twisted tensor product* $A \hat{\otimes} B$ as $A \hat{\otimes} B := T^{-1}(A \otimes B)T$.

Definition. Let $\mathcal{H}$ be a separable Hilbert space. We say $T \in L^2(\mathbb{R}^d, \mathrm{d}x; \mathcal{B}(\mathcal{H}))$ if $T : \mathbb{R}^d \ni x \mapsto T(x) \in \mathcal{B}(\mathcal{H})$ is a weakly measurable function such that

$$\|T\|_{L^2(\mathbb{R}^d, \mathrm{d}x; \mathcal{B}(\mathcal{H}))} := \left(\int \|T(x)\|^2_{B(\mathcal{H})} \,\mathrm{d}x \right)^{1/2} < \infty.$$

2.2 Fock and coherent representations

Here we describe some well known facts about bosonic Fock spaces and coherent representation of CCR (for more details we refer the reader, for instance, to [7] and [9]).

2.2.1 Bosonic space and creation/annihilation operators

Let $\mathfrak{h}$ be a Hilbert space. The *bosonic Fock* space over $\mathfrak{h}$ is the direct sum $\Gamma(\mathfrak{h}) := \bigoplus_{n=0}^\infty \bigotimes_{\mathrm{s}}^n \mathfrak{h}$ where $\bigotimes_{\mathrm{s}}^n \mathfrak{h}$ denote the symmetric n-th tensor power of $\mathfrak{h}$. The *number operator* N is defined as $\mathrm{N}|_{\bigotimes_{\mathrm{s}}^n \mathfrak{h}} = n\mathbb{1}$. If $h \in \mathfrak{h}$,

we define the *creation operator* $a^*(h)$ and the *annihilation operator* $a(h)$ by setting, for $u \in \bigotimes_{\mathrm{s}}^n \mathfrak{h}$,

$$a^*(h)u := \sqrt{n+1}\, u \otimes_{\mathrm{s}} h,$$
$$a(h)u := \sqrt{n}(h|\, u.$$

By $a^\sharp(h)$ we mean both $a^*(h)$ and $a(h)$.
If the one-particle space is $\mathfrak{h} = L^2(\mathbb{R}^d, \mathrm{d}k)$, then we can define the expressions $a(k)$, $a^*(k)$ by:

$$a(h) =: \int a(k)\bar{h}(k)\mathrm{d}k$$
$$a^*(h) =: \int a^*(k)h(k)\mathrm{d}k.$$

We define the *Segal field operators*:

$$\Phi(h) := \frac{1}{\sqrt{2}}(a^*(h) + a(h)).$$

Let $\Omega \in \bigotimes^0 \mathfrak{h}$ denote the *vacuum vector*. There exists a large class of representations of the CCR, called *g-coherent representations*, which are constructed by defining the new creation/annihilation operators a_g^*/a_g acting on $\Gamma(\mathfrak{h})$ as follows: let $\mathfrak{h}_0$ be a dense subspace of $\mathfrak{h}$ and $g \in \mathfrak{h}_0'$ (the dual of $\mathfrak{h}_0$); then we define:

$$\begin{aligned} a_g^*(h) &:= a^*(h) + \tfrac{1}{\sqrt{2}} < g, h > \\ a_g(h) &:= a(h) + \tfrac{1}{\sqrt{2}}\overline{< g, h >}, \end{aligned} \tag{2.3}$$

where $<\, ,\, >$ is the duality bracket.

The following fact is well known (see for instance [9, Theorem 3.2]):

Proposition 2.1. *(i) if $g \in \mathfrak{h}$, (2.3) can be rewritten as $a_g^\sharp(h) = e^{\Phi(-\mathrm{i}g)} a^\sharp(h) e^{\Phi(\mathrm{i}g)}$,*

(ii) if $g \notin \mathfrak{h}$, there exists no unitary operator U such that $a_g^\sharp(h) = U^* a^\sharp(h) U$.

In other words a g-coherent representation is unitarily equivalent to the Fock representation if and only if $g \in \mathfrak{h}$.

2.2.2 The operator $\mathrm{d}\Gamma$

If b is an operator on $\mathfrak{h}$, we define the operator $\mathrm{d}\Gamma(b) : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h})$ by

$$\mathrm{d}\Gamma(b)|_{\bigotimes_{\mathrm{s}}^n \mathfrak{h}} := \sum_{i=1}^{n} \underbrace{1\!\!1 \otimes \cdots \otimes 1\!\!1}_{i-1} \otimes b \otimes \underbrace{1\!\!1 \otimes \cdots \otimes 1\!\!1}_{n-i}.$$

If $g \in \mathfrak{h}$, we define $\mathrm{d}\Gamma_g(b) := e^{\Phi(-\mathrm{i}g)} \mathrm{d}\Gamma(b) e^{\Phi(\mathrm{i}g)}$ and one can compute that

$$\mathrm{d}\Gamma_g(b) = \mathrm{d}\Gamma(b) + \phi(bg) + \frac{1}{2}(g, bg), \tag{2.4}$$

provided $b \geq 0$ and $g \in D(b^{1/2})$. If $\mathfrak{h}_0 \subset D(b^{1/2})$ and $b^{1/2} : \mathfrak{h}_0 \to \mathfrak{h}_0$ then by duality $b^{1/2} : \mathfrak{h}_0' \to \mathfrak{h}_0'$. If $g \in \mathfrak{h}_0' \setminus \mathfrak{h}$ and $b^{1/2}g \in \mathfrak{h}$, we can make sense of the expression in the right hand side of (2.4) and define $\mathrm{d}\Gamma_g(b)$ in the same way.

2.2.3 The operators Γ and $\check{\Gamma}$

Let $\mathfrak{h}_i$, $i = 1, 2$ be two Hilbert spaces. If $q \in B(\mathfrak{h}_1, \mathfrak{h}_2)$, we define the operator $\Gamma(q) : \Gamma(\mathfrak{h}_1) \to \Gamma(\mathfrak{h}_2)$

$$\Gamma(q)|_{\bigotimes_s^n \mathfrak{h}_1} := q \otimes \cdots \otimes q.$$

Let j_1,$j_2 \in B(\mathfrak{h})$. We denote by $j = (j_1, j_2)$ the operator $j : \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h}$ defined by $jh := (j_1 h, j_2 h)$. We define the operator $\check{\Gamma}(j) : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h} \otimes \mathfrak{h})$ as:

$$\check{\Gamma}(j) := U\Gamma(j),$$

where $U : \Gamma(\mathfrak{h} \oplus \mathfrak{h}) \to \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h})$ is the *exponential map* defined by

$$U\Omega = \Omega \otimes \Omega \qquad Ua^\sharp(h_1 \oplus h_2) = (a^\sharp(h_1) \otimes 1\!\!1 + 1\!\!1 \otimes a^\sharp(h_2))U, \quad h_i \in \mathfrak{h}. \tag{2.5}$$

Assume j is isometric *i.e.* $j_1^* j_1 + j_2^* j_2 = 1$, then $\check{\Gamma}^*(j)\check{\Gamma}(j) = 1$. Moreover, if $j_i = j_i^*$, $i = 1, 2$, then using (2.5) one can easily check the following:

$$\check{\Gamma}(j)a^\sharp(h) = (a^\sharp(j_1 h) \otimes 1\!\!1 + 1\!\!1 \otimes a^\sharp(j_2 h))\check{\Gamma}(j), \tag{2.6}$$

$$\mathrm{d}\Gamma(b) = \check{\Gamma}^*(j)(\mathrm{d}\Gamma(b) \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(b))\check{\Gamma}(j) + \frac{1}{2}\mathrm{d}\Gamma(ad_{j_1}^2 b + ad_{j_2}^2 b), \tag{2.7}$$

where b is an operator on $\mathfrak{h}$.
We define the operator $\mathrm{N}^{\mathrm{ext}} : \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h})$ by

$$\mathrm{N}^{\mathrm{ext}} := \mathrm{N} \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{N}.$$

2.2.4 The Nelson Model

The Hilbert space $\mathcal{H}$ is $\mathcal{H} := \mathcal{K} \otimes \Gamma(\mathfrak{h})$ where $\mathcal{K}$ is the N-particle space $\mathcal{K} := \bigwedge_{j=1}^N L^2(\mathbb{R}^3, \mathrm{dx}_j)$ (the spin is neglected but the Fermi statistics is kept) and $\Gamma(\mathfrak{h})$ is the Fock space with $\mathfrak{h} = L^2(\mathbb{R}^3, \mathrm{d}k)$. To avoid confusion we denote the fermion position by $\mathrm{x} \in \mathbb{R}^3$, the boson position by $x \in \mathbb{R}^3$, $x := \mathrm{i}\nabla_k$ and the position of the system of N-fermions by $X \in \mathbb{R}^{3N}$, $X = (\mathrm{x}_1, \ldots, \mathrm{x}_N)$.
The Hamiltonian H_N is given by

$$H_N = K_N \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega) + \lambda\Phi(v). \tag{2.8}$$

where λ is a coupling constant. The operator K_N is given by (1.2) and we assume hypothesis *(I)* given in the introduction.
The operator ω is the operator of multiplication by the function $\omega(k)$. For the sake of simplicity we consider only the physical case $\omega(k) = |k|$.
The operator $\Phi(v)$ is given by

$$\Phi(v) = \frac{1}{\sqrt{2}} \int v^*(k)a(k) + v(k)a^*(k)\mathrm{d}k,$$

where $v : \mathbb{R}^3 \to \mathcal{B}(\mathcal{K})$ is defined by

$$v(k) := \sum_{j=1}^N \frac{e^{-\mathrm{i}k\mathrm{x}_j}}{\omega(k)^{1/2}}\rho(k)$$

where $\rho \in C_0^\infty(\mathrm{B}(0, \Lambda))$ with $\rho(-k) = \bar{\rho}(k)$ or equivalently $\check{\rho}$ real.
It is well known that this Hamiltonian is well defined and bounded from below (see for example [12]).

Notation: for simplicity we will drop the dependence on N everywhere, unless it needs to be specified.

2.2.5 Infrared cutoff Hamiltonians

We will need *infrared cutoff Hamiltonians* H_μ for $\mu > 0$. We define

$$H_\mu := K \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega) + \lambda\Phi(v_\mu), \tag{2.9}$$

where $v_\mu(k) := \chi_\mu(k)v(k)$ with $\chi_\mu(k) := \chi(\frac{k}{\mu})$ and $\chi \in C^\infty(\mathbb{R}^3)$ such that $\chi \equiv 1$ for $|k| > 2$ and $\chi \equiv 0$ for $|k| < 1$, $\chi_\mu(-k) = \bar{\chi}(k)$.
We will also need another cutoff Hamiltonian $\hat{H}_\mu : \mathcal{H} \to \mathcal{H}$ defined by:

$$\hat{H}_\mu := K \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega_\mu) + \lambda\Phi(v_\mu), \tag{2.10}$$

where $\omega_\mu \in C^\infty(\mathbb{R}^3)$, $\omega_\mu(k) := \mu\ \omega_1(\frac{k}{\mu})$ and $\omega_1(k) = \omega_1(|k|)$ is a smooth function, increasing with respect to $|k|$, equal to ω on $\{1 < |k|\}$ and equal to $1-\delta$ on $\{|k| < 1-\delta\}$, $\delta << 1$. Note that $\omega_\mu \geq \tilde{\mu}$ where $\tilde{\mu} = (1-\delta)\mu$.

2.2.6 Nelson model in a coherent representation

We consider the Nelson model in a g-coherent representation, the same originally considered by Arai in [1].

Choosing $g = -\lambda N \frac{\rho(k)}{\omega(k)^{3/2}}$, and using (2.4) and (2.3), the new Hamiltonian becomes

$$H_N^{\mathrm{ren}} = K_N^{\mathrm{ren}} \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega) + \lambda\phi(v^{\mathrm{ren}}) \tag{2.11}$$

where

$$v^{\mathrm{ren}} = v - \omega g, \qquad K_N^{\mathrm{ren}} = K_N + W(X),$$

with

$$W(X) = -\lambda^2 N \sum_{i=1}^N w(\mathrm{x}_i) + \lambda^2 \frac{N^2}{2} w(0), \qquad w(\mathrm{x}) = \int_{\mathbb{R}^3} e^{\mathrm{i}k\mathrm{x}} \frac{|\rho|^2(k)}{\omega^2(k)}\ \mathrm{d}k.$$

Note that $w(\mathrm{x}) = O(|\mathrm{x}|^{-1})$ as $|\mathrm{x}| \to \infty$ and that $w(\mathrm{x}) = \frac{1}{4\pi}\check{\rho} * \check{\rho} * \frac{1}{|\mathrm{x}|}$where $\check{\rho}$ is the inverse Fourier transform of ρ.
For $\mu > 0$ we use $g_\mu = -\lambda N \frac{\rho(k)}{\omega(k)^{3/2}} \chi_\mu(k)$ and we obtain

$$H_{N,\mu}^{\mathrm{ren}} = K_{N,\mu}^{\mathrm{ren}} \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega) + \lambda\Phi(v_\mu^{\mathrm{ren}}) \tag{2.12}$$

with

$$v_\mu^{\mathrm{ren}} = v_\mu - \omega g_\mu,$$

and $K_{N,\mu}^{\mathrm{ren}}$ is the same as before replacing w by w_μ with

$$w_\mu(\mathrm{x}) := \int_{\mathbb{R}^3} e^{\mathrm{i}k\mathrm{x}} \frac{|\rho|^2(k)}{\omega^2(k)} |\chi_\mu|^2(k)\mathrm{d}k,$$

and W by W_μ consequently.

Remark 2.1. Because of Proposition 2.1, H_μ and H_μ^{ren} are unitarily equivalent while H and H^{ren} are not.

3 Binding condition

Let A_N be a family of self-adjoint operators on $\mathcal{H}$ depending on $N \in \mathbb{N}$ and A_N^0 the corresponding family of their translation invariant part. Assume that all the operators are bounded from below. For A a self-adjoint operator let us denote $E(A) := \inf \sigma(A)$. Then we can define the *ionization threshold energy* of A_N as

$$\tau(A_N) := \inf_{0<N'\leq N} \{E(A_{N'}^0) + E(A_{N-N'})\}.$$

The binding condition then is

$$(B) \qquad E(A_N) < \tau(A_N).$$

From a physical point of view this is a minimal condition to require on the external potential to be binding. From a mathematical point of view another energy can be considered. This is the energy below which exponential decay can be proved in a quite general way, as it was done in [14], so we can call it *localization energy* $\Sigma(A)$. Let us define

$$D_R := \{\psi \in \mathcal{H} \mid \psi(X) = 0 \text{ if } |X| < R\}.$$

We define

$$\Sigma_R(A) := \inf_{\psi \in D_R \cap D(A),\, ||\psi||=1} (\psi, A\psi), \tag{3.13}$$

and

$$\Sigma(A) := \lim_{R\to\infty} \Sigma_R(A). \tag{3.14}$$

In [14] it is also proved that for the standard model of non-relativistic QED the two energies are the same; this is also true in our case as we will explain in subsection 4.2.

A key observation in the proof of exponential decay is that both H_μ and $\hat{H}_\mu$ have the same ionization and localization energy. This is stated in the following Lemma.

Lemma 3.1. *For every $\mu > 0$, $\Sigma(H_\mu) = \Sigma(\hat{H}_\mu)$ and $\tau(H_\mu) = \tau(\hat{H}_\mu)$.*

Proof. Set $\mathfrak{h}_\mu := L^2(\{|k| \le \mu\})$, $\mathbb{C}_\mu := \otimes^0 \mathfrak{h}_\mu$, $\mathcal{H}_{\text{int}} := \mathcal{K} \otimes \Gamma(\mathfrak{h}_\mu^\perp)$. Since $\mathfrak{h} = \mathfrak{h}_\mu \oplus \mathfrak{h}_\mu^\perp$, we can identify $\mathcal{H}$ and $\mathcal{H}_{\text{int}} \otimes \Gamma(\mathfrak{h}_\mu)$ by exponential map. One can observe $\inf \sigma(H_\mu) = \inf \sigma(H_\mu|_{\mathcal{H}_{\text{int}}\otimes\mathbb{C}_\mu})$, $\inf \sigma(\hat{H}_\mu) = \inf \sigma(\hat{H}_\mu|_{\mathcal{H}_{\text{int}}\otimes\mathbb{C}_\mu})$. Since $H_\mu|_{\mathcal{H}_{\text{int}}\otimes\mathbb{C}_\mu} = \hat{H}_\mu|_{\mathcal{H}_{\text{int}}\otimes\mathbb{C}_\mu}$, the lemma follows. $\square$

Thanks to the above Lemma, we can introduce the following notation:

$$\Sigma_{R,\,\mu} := \begin{cases} \Sigma_R(\hat{H}_\mu) = \Sigma_R(H_\mu) & \text{for } \mu > 0, \\ \Sigma_R(H) & \text{for } \mu = 0, \end{cases} \qquad \tau_\mu := \begin{cases} \tau(\hat{H}_\mu) = \tau(H_\mu) & \text{for } \mu > 0, \\ \tau(H) & \text{for } \mu = 0, \end{cases}$$

$\Sigma_\mu := \lim_{R\to\infty} \Sigma_{R,\,\mu}$.

Remark 3.1. Since H_μ and H_μ^{ren} converge in the norm resolvent sense to H and H^{ren} respectively (see [12, Lemma A.2]), and H_μ and H_μ^{ren} are unitarily equivalent by Proposition 2.1. Hence $E(\hat{H}_\mu) = E(H_\mu) = E(H_\mu^{\text{ren}})$ and $E(H) = E(H^{\text{ren}})$.

Thanks to the above remark we can give the following definition.
Definition. We define

$$E_\mu := E(\hat{H}_\mu) = E(H_\mu) = E(H_\mu^{\text{ren}}),$$
$$E := E(H) = E(H^{\text{ren}}).$$

It is easy to prove that if the binding condition is satisfied when neglecting the field, then it still holds for λ small enough.

Proposition 3.1. *Assume $E(K_N) < \tau(K_N)$. Then $E(H_N) < \tau(H_N)$ for λ small enough.*

Proof. Set $H_0 := K_N \otimes \mathbb{1} + \mathbb{1} \otimes d\Gamma(\omega)$ and $H_\lambda := H$. Since $\Phi(v)$ is H_0-bounded as operator (see [19, Lemma 2]) then clearly $H_\lambda \to H_0$ as $\lambda \to 0$ in the norm resolvent sense, which implies $E(H_\lambda) \to E(H_0)$. Now $E(H_0) = E(K_N)$ and $\tau(H_0) = \tau(K_N)$. $\square$

Remark 3.2. If $v(\mathrm{x}) = -\frac{Z}{|\mathrm{x}|}$ and $u(\mathrm{x}) = \frac{1}{|\mathrm{x}|}$, it is well known that $E(K_N) < E(K_{N-1})$ if $N < Z + 1$ (see [22]).

It is also possible to prove that at least one particle must be bounded. This proves *(B)* for $N = 1$.

Proposition 3.2 (binding of at least one particle)**.** *Assume (I) holds and that the operator* $-\frac{1}{2}\triangle + v$ *admits an eigenvalue of energy* $-e_0 < 0$*. Then* $E(H_1) < \tau(H_1)$*.*
If in addition $v(\mathrm{x}) \leq 0$ *for all* x*, then* $E(H_N) \leq E(H_N^0) - e_0$*.*

Proof. The proof is the same as in [13, Theorem 3.1], and is therefore omitted. □

We remark that binding without mass implies binding with mass.

Proposition 3.3. *Assume* $E < \tau_0$*. Then* $E_\mu < \tau_\mu$ *for* μ *small enough.*

Proof. As $\mu \to 0$, H_μ converges in the norm resolvent sense to H (see [12, Lemma A.2]). Hence as $\mu \to 0$, E_μ converges to E and τ_μ converges to τ_0.

4 Exponential decay

4.1 Localization Lemma

Here we state a key Lemma about boson localization needed in the next subsection to prove exponential decay.

Remark 4.1. The next proposition is true for $\hat{H}_\mu$ but not for H_μ. This is one of the reason why the infrared regularization $\hat{H}_\mu$ was introduced.

Lemma 4.1. *Let* $\hat{H}_\mu$ *be the Hamiltonian defined in (2.8). Then*

$$\hat{H}_\mu \geq \tau_\mu - f(\mu) o\left(R^0\right)\left(\hat{H}_\mu + C\right) \text{ on } D_R,$$

where $f(\mu) := \frac{\ln^{1/2}\mu}{\mu}$ *and* $D_R := \{\psi \in \mathcal{H} \,|\, \psi(X) = 0 \text{ if } |X| < R\}$*.*

Proof. The proof is the same (simplified) as in [13, Corollary A.2] and [14, Theorem 9]. Only the dependence on the error on the infrared parameter is different, and the estimate needed in our case is given by the next Lemma. Notice in [13] this dependence was not explicitly considered and it is not uniform in μ as stated. This gap left in the proof was filled in [14]. Since their proof is long and the reader could get lost, we provide a sketch of the proof in our case in Appendix A. □

Lemma 4.2. *Fix* $X = (\mathrm{x}_1, \ldots, \mathrm{x}_2) \in \mathbb{R}^{3N}$*. Let* $j \in C^\infty(\mathbb{R}^3)$ *be a function such that*

(i) $0 \leq j \leq 1$*,*

(ii) $\operatorname{supp} j_R \in \{x \,|\, |x - \mathrm{x}_j| > R\}$ *(where* $j_R(x) := j(\frac{x}{R})$*),*

Then $\|j_R(x)\hat{v}_\mu(x - \mathrm{x}_j)\|_{L^2(\mathbb{R}^3, \mathrm{d}x)} = O(R^{-1})O(ln^{1/2}\mu)$ *uniformly in* X*.*

Proof. Here $\hat{v}_\mu(x - \mathrm{x}_j)$ is $\mathcal{F}v_\mu(k, \mathrm{x}_j)$, where $\mathcal{F}$ is the Fourier transform with respect to the variable k. In other words we use the unitary equivalence of the space $\mathfrak{h}$ and $L^2(\mathbb{R}^3, \mathrm{d}x) =: \mathfrak{h}_x$ given by the Fourier transform. Since $j \leq 1$ and $\operatorname{supp} j_R \subset \{x \,|\, |x - \mathrm{x}_j| > R\}$, there exists a function $F \in C_0^\infty(\mathbb{R}^3)$, with $F(0) = 1$ such that

$$\|j_R(x)\hat{v}_\mu(x - \mathrm{x}_j)\|_{\mathfrak{h}_x} \leq \left\|(1 - F(\frac{x}{R}))\hat{v}_\mu(x)\right\|_{\mathfrak{h}_x} = \left\|(1 - F(\frac{D_k}{R}))v_\mu(k, 0)\right\|_{\mathfrak{h}},$$

since $\hat{v}_\mu(x) = \mathcal{F}v_\mu(k, 0)$. By standard pseudodifferential calculus

$$\|(1 - F(\frac{D_k}{R}))v_\mu(k, 0)\|_{\mathfrak{h}} \leq \frac{1}{R}\|\nabla v_\mu(k, 0)\|_{\mathfrak{h}} = \frac{1}{R}O(\ln^{1/2}\mu)$$

as one can easily compute. □

4.2 Exponential Decay

In this subsection we prove uniform exponential decay for states of energy lower than the ionization energy τ_μ for all $\mu \geq 0$. The proof consist in two parts. First we prove localization below Σ_μ; this can be done in a more general framework (Prop. 4.1). This argument and its proof are the same as in [14], we only take care of checking that the estimates are uniform in the infrared parameter μ, which is left to the reader in [14]. Secondly, we prove $\Sigma_\mu = \tau_\mu$ for all $\mu \geq 0$. This is also as in [14]. Remark that in [14] the two different infrared regularizations are used. Collecting the results, one obtain exponential decay for our model (Corollary 4.2).

Proposition 4.1 (uniform exponential decay)**.** *Let $\mathcal{H} := L^2(X) \otimes \mathcal{H}_1$, where $\mathcal{H}_1$ is an auxiliary Hilbert space. Let $H(\mu)$ for $0 \leq \mu \leq 1$ be a family of self-adjoint operators on $\mathcal{H}$. Assume there exists D dense in $\mathcal{H}_1$ such that $C_0^\infty(X) \otimes D$ is a core for $H(\mu)$. We assume also*

(i) *the IMS localization formula is satisfied i.e.*

$$H(\mu) = \sum_{i=1}^{2} j_i H(\mu) j_i - \sum_{i=1}^{2} |\nabla j_{i,R}|^2, \text{ if } j_1^2(X) + j_2^2(X) = 1,$$

(ii) *if $g \in C^\infty(X, \mathbb{R})$ then $e^g H(\mu) e^{-g} + e^{-g} H(\mu) e^g = 2H - 2|\nabla g|^2$,*

as quadratic forms on $C_0^\infty(X) \otimes D$.
Let us denote $\Sigma_R(\mu) := \Sigma_R(H(\mu))$, $\Sigma(\mu) := \Sigma(H(\mu))$ and $E(\mu) := \inf \sigma(H(\mu))$. We suppose

(iii) *$E(\mu) \to E(0)$,*

(iv) *$\Sigma_R(\mu) \to \Sigma_R(0)$ uniformly in R,*

as $\mu \to 0$.
Fix a function $g \in C^\infty(X, \mathbb{R})$ such that $|\nabla g|^2 \leq 1$.
If $\Sigma(0) - E(0) = \delta_0 > 0$ then for all $f \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp} f \subset]-\infty, \Sigma(0)[$ there exists μ_0 and β_0 (depending only on f) such that for all $\mu \leq \mu_0$, $\beta < \beta_0$

$$\left\| e^{\beta g} f(H(\mu)) \right\| \leq C \text{ uniformly in } \mu.$$

Proof. Since $\operatorname{supp} f \subset]-\infty, \Sigma(0)[$ we can assume that $\operatorname{supp} f \subset]-\infty, \Sigma(0) - \alpha]$, $\alpha > 0$. Since $E(\mu) \to E(0)$ and $\Sigma(\mu) \to \Sigma(0)$, there exists μ_0 such that for all $\mu \leq \mu_0$

$$|\Sigma(\mu) - \Sigma(0)| < \frac{\alpha}{4}, \tag{4.15}$$
$$|E(\mu) - E(0)| < \frac{\alpha}{4}, \qquad |\Sigma(\mu) - E(\mu)| > \delta_0 - \frac{\alpha}{2}.$$

By (4.15) $\operatorname{supp} f \subset]-\infty, \Sigma(\mu)[$ for all $\mu \leq \mu_0$.

Let us denote

$$H_{R,\mu} := H(\mu) + (\Sigma_R(\mu) - E(\mu))\chi_R$$

where χ is a smoothed characteristic function of the unit ball. Take $j_1, j_2 \in C^\infty(X)$ such that $j_1^2 + j_2^2 = 1$, $j_1 = 1$ on $\mathrm{B}(0, \frac{1}{2})$ and $j_1 = 0$ outside $\mathrm{B}(0,1)$. By the IMS localization formula $H_{R,\mu} = \sum_{i=1}^{2} j_{i,R} H_{R,\mu} j_{i,R} - \sum_{i=1}^{2} |\nabla j_{i,R}|^2$. By the definition of $H_{R,\mu}$ and since $\chi_R = 1$ on $\operatorname{supp} j_{1,R}$

$$\begin{aligned} &j_{1,R} H_{R,\mu} j_{1,R} = j_{1,R}(H(\mu) + \Sigma_R(\mu) - E(\mu)) j_{1,R} \geq \Sigma_R(\mu) j_{1,R}^2, \\ &j_{2,R} H_{R,\mu} j_{2,R} \geq \Sigma_R(\mu) j_{2,R}^2. \end{aligned}$$

Hence

$$H_{R,\mu} \geq \Sigma_R(\mu)(j_1^2 + j_2^2) - \sum_{i=1}^{2} |\nabla j_{i,R}|^2 \geq \Sigma_R(\mu) - CR^{-2},$$

uniformly in μ. For $R \geq R_0$ and $\mu \leq \mu_0$

$$\Sigma_R(\mu) - \frac{C}{R^2} \geq \Sigma_R(\mu) - \frac{\alpha}{4} \geq \Sigma(0) - \frac{\alpha}{2}.$$

If $\lambda \in \operatorname{supp} f$, then $\lambda \leq \Sigma(0) - \alpha = E(0) + \delta_0 - \alpha$. We have $\Sigma(0) - \frac{\alpha}{2} > E(0) + \delta_0 - \frac{\alpha}{4}$. Hence for $R \geq R_0$, $f(H_{R,\mu}) = 0$ for $\mu \leq \mu_0$.
We want now to show (for any *fixed* $R \geq R_0$)

$$e^{\beta g} f(H(\mu)) = e^{\beta g}(f(H(\mu)) - f(H_{R,\mu})) \text{ is bounded uniformly in } \mu.$$

By Theorem B.1 we can write

$$\begin{aligned} e^{\beta g} \quad & f(H(\mu)) = e^{\beta g}(f(H(\mu)) - f(H_{R,\mu})) = \\ &= \tfrac{1}{\mathrm{i}\pi} \int_{\mathbb{C}} \tfrac{\partial \tilde{f}}{\partial \bar{z}}(z) e^{\beta g}(z - H_{R,\mu})^{-1} e^{\beta g} e^{-\beta g}(\Sigma_R(\mu) - E(\mu))\chi_R(z - H(\mu))^{-1} dz \wedge d\overline{z}. \end{aligned}$$

Since $\operatorname{supp} \tilde{f}$ is compact, it suffices to estimate the integrand:

$$\begin{aligned} & \left\| e^{\beta g}(z - H_{R,\mu})^{-1} e^{\beta g} e^{-\beta g}(\Sigma_R(\mu) - E(\mu))\chi_R(z - H(\mu))^{-1} \right\| \leq \\ & \leq \left\| e^{\beta g}(z - H_{R,\mu})^{-1} e^{-\beta g} \right\| \left\| e^{\beta g}\chi_R \right\|_\infty |\Sigma_R(\mu) - E(\mu)| \left\| (z - H(\mu))^{-1} \right\|. \end{aligned}$$

We have $\left\| e^{\beta g}\chi_R \right\|_\infty \leq e^{\beta g_R}$ where $g_R := \sup_{\{|X|<R\}} g(X)$, $\quad \Sigma_R(\mu) - E(\mu) \leq \Sigma(\mu) - E(\mu)$, $\left\| (z - H)^{-1} \right\| \leq |\mathrm{Im} z|^{-1}$. It remains to estimate $\left\| e^{\beta g}(z - H_{R,\,\mu})^{-1} e^{-\beta g} \right\|$. First we notice that

$$e^{\beta g}(z - H_{R,\,\mu})^{-1} e^{-\beta g} = (z - e^{\beta g} H_{R,\,\mu} e^{-\beta g})^{-1}$$

and that

$$\begin{aligned} & \mathrm{Re}(e^{\beta g} H_{R,\,\mu} e^{-\beta g} - z) = H_{R,\,\mu} - \beta^2 |\nabla g|^2 - \mathrm{Re} z \\ & \geq \Sigma(\mu) - \tfrac{\alpha}{4} - \beta^2 - (E(0) + \delta_0 - \alpha) \\ & \geq \Sigma(0) - \tfrac{\alpha}{2} - \beta^2 - (E(0) + \delta_0 - \alpha) \geq \tfrac{\alpha}{2} - \beta^2. \end{aligned}$$

This implies for $\beta^2 < \frac{\alpha}{2}$ that $\left\| e^{\beta g}(z - H_{\mu,\,R})^{-1} e^{-\beta g} \right\| \leq \frac{1}{\frac{\alpha}{2} - \beta^2}$ uniformly in μ.
Collecting all the estimates we obtain

$$\left\| e^{\beta g} f(H(\mu)) \right\| \leq \frac{(\Sigma(0) - E(0) + \frac{\alpha}{2}) e^{\beta g_R}}{\frac{\alpha}{2} - \beta^2}.$$

□

Lemma 4.3. *(i) $\Sigma_\mu < \infty$ if and only if $\Sigma_0 < \infty$ and in this case there exists a constant C such that $|\Sigma_{R,\,\mu} - \Sigma_{R,\,0}| < C\mu^{1/2}$ uniformly in R,*

(ii) there exists a constant C such that $|\tau_\mu - \tau_0| < C\mu^{1/2}$.

Proof. The proof is the same as in [14, Proposition 5] just noting that the estimates concerning *(i)* are uniform in R. □

Proposition 4.2. *For every $\mu \geq 0$, $\Sigma_\mu = \tau_\mu$.*

Proof. As in [14, Theorem 6], using Lemma 4.1. □

Corollary 4.1. *Assume (B) and (I). Let*

$$H(\mu) := \begin{cases} H_\mu \quad or \quad \hat{H}_\mu \quad for \quad \mu > 0, \\ H \quad for \quad \mu = 0. \end{cases}$$

Then then for all $f \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp} f \subset]-\infty, \tau_0[$ *there exists a μ_0 and β_0 (depending only on f) such that for all $\mu \leq \mu_0$, $\beta < \beta_0$*

$$\left\| e^{\beta |X|} f(H(\mu)) \right\| \leq C \textit{ uniformly in } \mu.$$

Proof. We apply Proposition 4.1 with $\mathcal{H}_1 = \Gamma(\mathfrak{h})$. Hypothesis (i) and (ii) are clearly verified by direct computation, hypothesis (iii)is true since both H_μ and $\hat{H}_\mu$ converges in the norm resolvent sense to H (see [12, Lemma A.2]) and (iv) follows from Lemma 4.3. By *(B)* $\Sigma_0 - E = \delta_0 > 0$. Choose $g_\varepsilon := \frac{\langle X \rangle}{1+\varepsilon\langle X \rangle}$, then $|\nabla g_\varepsilon| \leq 1$ uniformly in ε and $\sup_{\{|X|<R\}} g_\varepsilon \leq R$ uniformly in ε. Hence by Proposition 4.1, there exists a μ_0 such that for all $\mu \leq \mu_0$, $\beta \leq \beta_0$, $\left\| e^{\beta|X|} f(H(\mu)) \right\| \leq C$ uniformly in μ. □

5 Proof of theorem 1.1

We can now prove the non-existence of the ground state in the Fock representation. We use the following Lemma from [9, Lemma 2.6].

Lemma 5.1. *Let $\psi \in \mathcal{H}$ be such that $\int_{\mathbb{R}^3} \|a(k)\psi + h(k)\psi\|^2_{\mathcal{H}} dk < \infty$ where $k \to h(k) \in \mathbb{C}$ is a mesurable function and $\int_{\mathbb{R}^3} |h(k)|^2 \mathrm{d}k = \infty$.*
Then $\psi = 0$.

Proof of Theorem 1.1. Suppose there exists ψ such that $H\psi = E\psi$. We want to prove $\psi \equiv 0$. By *(B)*, there exists a function $f \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\operatorname{supp} f \subset]-\infty, \tau_0[$ and $f \equiv 1$ on an interval $[E, E+\delta]$, $\delta > 0$. By Corollary 4.1, $|X|f(H)$ is bounded, so $|X| = |X|f(H)\psi$ belongs to $\mathcal{H}$.

By the pullthrough formula (as an identity on $L^2_{\mathrm{loc}}(\mathbb{R}^3 \backslash \{0\}, \mathrm{d}k; \mathcal{H})$) we have

$$-a(k)\psi = (H - E + \omega(k))^{-1} \sum_{i=1}^{N} e^{-\mathrm{i}k\mathrm{x}_i} \frac{\rho(k)}{\omega(k)^{\frac{1}{2}}} \psi.$$

Writing $e^{-\mathrm{i}k\mathrm{x}_i}$ as $e^{-\mathrm{i}k\mathrm{x}_i} = 1 + r(\mathrm{x}_i, k)$ with $|r(k, \mathrm{x})| \leq |k||\mathrm{x}|$, the former expression becomes:

$$-a(k)\psi = m(k)\psi + h(k)\psi$$

where

$$m(k) := \sum_{i=1}^{N} (H - E + \omega(k))^{-1} r(k, \mathrm{x}_i) \frac{\rho(k)}{\omega(k)^{\frac{1}{2}}}, \qquad h(k) := N \frac{\rho(k)}{\omega(k)^{\frac{3}{2}}}.$$

Since $|X|\psi \in \mathcal{H}$, clearly $|\mathrm{x}_i|\psi \in \mathcal{H}$ and $\|r(k, \mathrm{x}_i)\psi\| \leq |k| \, \||\mathrm{x}_i|\psi\|$, which implies $m(k) \in L^2(\mathbb{R}^3, \mathrm{d}k)$. So we have $\int \|m(k)\|^2_{\mathcal{H}} \mathrm{d}k = \int \|(a(k) + h(k))\psi\|^2_{\mathcal{H}} \mathrm{d}k < \infty$. Since $h(k) \notin L^2(\mathbb{R}^3, \mathrm{d}k)$, then ψ must be 0 by Lemma 5.1. □

6 Proof of theorem 1.2

6.1 Existence of ground state in the massive case

We adapt the proof in [12]. We want to prove the existence of a ground state for H_μ^{ren}. Because of Proposition 2.1, H_μ^{ren} is unitarily equivalent to H_μ; by Lemma 6.1, H_μ admits a ground state if and only if $\hat{H}_\mu$ does. We prove the existence of a ground state for $\hat{H}_\mu$ by showing that there is a spectral gap.

Lemma 6.1. *$\hat{H}_\mu$ admits a ground state if and only if H_μ admits a ground state.*

Proof. See [12, Lemma 3.2] . □

Lemma 6.2. *Let $\chi \in C_0^\infty(\mathbb{R})$ be a function such that $\operatorname{supp}\chi \subset]-\infty, \tau_\mu[$. Let $j \in C_0^\infty(\mathbb{R}^3)$ be such that $j = 1$ on $B(0,1)$ and $j = 0$ outside $B(0,2)$. Assume (I) and (B).Then the operator $\Gamma(j_R^2)\chi(\hat{H}_\mu)$ is compact.*

Proof. Let us denote $C := \Gamma(j_R^2)\chi(\hat{H}_\mu)$. It suffices to prove that C^*C is compact. We have

$$\begin{aligned}&C^*C = \chi_P(\hat{H}_\mu)\Gamma(j_R^4)\chi_P(\hat{H}_\mu) = \\ &= \chi_P(\hat{H}_\mu)(-\triangle_{\mathrm{x}}^{1/2} + |X| + 1)(-\triangle_{\mathrm{x}}^{1/2} + |X| + 1)^{-1} \times \\ &\times \Gamma(j_R^4)\ (\mathrm{d}\Gamma(\omega_\mu) + 1)^{-1}(\mathrm{d}\Gamma(\omega_\mu) + 1)\chi(\hat{H}_\mu).\end{aligned}$$

The operator $\chi(\hat{H}_\mu)(-\triangle_{\mathrm{x}}^{1/2})$ is bounded since $D(|\hat{H}_\mu|^{1/2}){=}D(|H_0|^{1/2})$ where $H_0 = -\frac{1}{2}\triangle_{\mathrm{x}} \otimes \mathbb{1}_{\Gamma(\mathfrak{h})} + \mathbb{1}_{\mathcal{K}} \otimes \mathrm{d}\Gamma(\omega)$. Since $\operatorname{supp}\chi \subset]-\infty, \tau_\mu[$, by Corollary 4.1 also $\chi(\hat{H}_\mu)|X|$ is bounded. Hence the operator $B_1 := \chi(\hat{H}_\mu)(-\triangle_{\mathrm{x}}^{1/2} + |X| + 1)$ is bounded on $\mathcal{K} \otimes \Gamma(\mathfrak{h})$.
Moreover $K_1 := (-\triangle_{\mathrm{x}}^{1/2} + |X| + 1)^{-1}$ is compact on $\mathcal{K}$, $K_2 := \Gamma(j_R^4)(\mathrm{d}\Gamma(\omega_\mu) + 1)^{-1}$ is compact on $\Gamma(\mathfrak{h})$, $B_2 := (\mathrm{d}\Gamma(\omega_\mu) + 1)\chi(\hat{H}_\mu)$ is bounded on $\mathcal{K} \otimes \Gamma(\mathfrak{h})$.
This implies that operator $C^*C = B_1(K_1 \otimes K_2)B_2$ is compact on $\mathcal{K} \otimes \Gamma(\mathfrak{h})$. □

Lemma 6.3. *Let $j := (j_0, j_\infty)$ where $j_0 = 1$ on $B(0,1)$, $j_0 = 0$ outside $B(0,2)$ and j_∞ such that $j_0^2 + j_\infty^2 = 1$. Let $H^{\mathrm{ext}} := \hat{H}_\mu \otimes \mathbb{1}_{\Gamma(\mathfrak{h})} + \mathbb{1}_{\mathcal{K}} \otimes \mathrm{d}\Gamma(\omega_\mu)$. Then*

$$\chi(H^{\mathrm{ext}})\check{\Gamma}(j_R) - \check{\Gamma}(j_R)\hat{H}_\mu = O(R^{-1})$$

Proof. By Theorem B.1

$$\begin{aligned}&\chi(H^{\mathrm{ext}})\check{\Gamma}(j_R) - \check{\Gamma}(j_R)\hat{H}_\mu = \\ &= \tfrac{1}{\mathrm{i}\pi}\int_{\mathbb{C}} \tfrac{\partial\tilde{\chi}}{\partial\bar{z}}(z)\check{\Gamma}(j_R)(z - \hat{H}_\mu)^{-1} - (z - H^{\mathrm{ext}})^{-1}\check{\Gamma}(j_R) dz \wedge d\overline{z} \\ &= \tfrac{1}{\mathrm{i}\pi}\int_{\mathbb{C}} \tfrac{\partial\tilde{\chi}}{\partial\bar{z}}(z)(z - \hat{H}_\mu)^{-1}(\check{\Gamma}(j_R)\hat{H}_\mu - H^{\mathrm{ext}}\check{\Gamma}(j_R))(z - H^{\mathrm{ext}})^{-1} dz \wedge d\overline{z}.\end{aligned}$$

Since $\operatorname{supp}\tilde{\chi}$ is compact it suffices to prove that $H^{\mathrm{ext}}\check{\Gamma}(j_R) - \check{\Gamma}(j_R)\hat{H}_\mu = O(R^{-1})(\mathrm{N}^{\mathrm{ext}}+1)$, which is equivalent to prove that $\hat{H}_\mu - \check{\Gamma}^*(j_R)H^{\mathrm{ext}}\check{\Gamma}(j_R) = O(R^{-1})(\mathrm{N}+1)$.
We have

$$\begin{aligned}&H_\mu - \check{\Gamma}^*(j_R)H^{\mathrm{ext}}\check{\Gamma}(j_R) = \mathrm{d}\Gamma(\omega_\mu) - \check{\Gamma}^*(j_R)\mathrm{d}\Gamma^{\mathrm{ext}}(\omega_\mu)\check{\Gamma}(j_R) \\ &\quad + \lambda\left(\Phi(v_\mu) - \check{\Gamma}^*(j_R)\Phi(v_\mu) \otimes \mathbb{1}_{\Gamma(\mathfrak{h})}\check{\Gamma}(j_R)\right).\end{aligned}$$

Now using (2.6)

$$\begin{aligned}&\mathrm{d}\Gamma(\omega_\mu) - \check{\Gamma}^*(j)\mathrm{d}\Gamma^{\mathrm{ext}}(\omega_\mu)\check{\Gamma}(j) = \mathrm{d}\Gamma(ad^2_{j_{0,\,R}}\omega_\mu + ad^2_{j_{\infty,\,R}}\omega_\mu) \\ &\quad \le \|ad^2_{j_{0,\,R}}\omega_\mu + ad^2_{j_{\infty,\,R}}\omega_\mu\|_{\mathfrak{h}}(N+1) \le O(R^{-1})(\mathrm{N}+1)\end{aligned}$$

and using (2.7)

$$\begin{aligned}&\Phi(v_\mu) - \check{\Gamma}^*(j_R)\Phi(v_\mu) \otimes \mathbb{1}_{\Gamma(\mathfrak{h})}\check{\Gamma}(j_R) = \\ &= \check{\Gamma}^*(j_R)\left(\phi((j_{0,\,R} - 1)v_\mu) \otimes \mathbb{1}_{\Gamma(\mathfrak{h})} + \mathbb{1}_{\Gamma(\mathfrak{h})}\hat{\otimes}\,\phi(j_{\infty,\,R}v_\mu))\right)\check{\Gamma}(j_R) \\ &\le (\|(j_{0,\,R} - 1)v_\mu\|_{\mathfrak{h}} + \|j_{\infty,\,R}v_\mu\|_{\mathfrak{h}})(\mathrm{N}+1) \le O(R^{-1})(\mathrm{N}+1)\end{aligned}$$

because of Lemma 4.2. □

Theorem 6.1 (Existence of spectral gap for $\hat{H}_\mu$). *Let $\hat{H}_\mu$ be the Hamiltonian defined in 2.10. Assume (I) and (B). Then $\sigma_{\mathrm{ess}}(\hat{H}_\mu) \subset [G_\mu, +\infty[$, where $G_\mu = min\{E_\mu + \tilde{\mu}, \tau_\mu\}$ with $\tilde{\mu} = \mu(1-\delta)$, $\delta << 1$. Consequently $\hat{H}_\mu$, H_μ and H_μ^{ren} admit a ground state.*

Proof. As in [12, Theorem 4.1], using Lemma 7.2 (instead of [12, Lemma 4.2]). □

6.2 Existence of a ground state in the massless case

Let ψ_μ be a ground state for H_μ^{ren}. We will prove Theorem 1.2 by showing that H^{ren} admits a ground state as limit of ψ_μ for $\mu \to 0$.

As mentioned in the introduction, the proofs in the confined case make use of the compactness of the operator $(K_N^{\mathrm{ren}} + \mathrm{i})^{-1}$, which does not hold anymore. Instead, we will use the localization in the fermion variables to control directly the behaviour as $k \to 0$ of $\|v^{\mathrm{ren}}(k)\psi_\mu\|_{\mathcal{H}}$. The facts we need are collected in the following Lemma.

Lemma 6.4. *We have*

$$\|v^{\mathrm{ren}}(k)\langle X\rangle^{-1}\|_{\mathcal{B}(\mathcal{K})} = \|\sum_{i=1}^{N} \frac{(e^{-\mathrm{i}k\mathrm{x}_i} - 1)\langle X\rangle^{-1}\rho(k)}{\omega(k)^{1/2}}\|_{\mathcal{B}(\mathcal{K})} \sim |k|^{1/2}. \tag{6.16}$$

Assume (B) and (I). Then for all $N \in \mathbb{N}$, and μ small enough

$$(\psi_\mu, \langle X\rangle^N \psi_\mu) \leq C, \ \textit{uniformly in } \mu > 0. \tag{6.17}$$

Moreover

$$\int \frac{\|v^{\mathrm{ren}}\psi_\mu\|^2_{\mathcal{H}}}{\omega(k)^\alpha}\mathrm{d}k \leq C \ \textit{uniformly in } \mu \textit{ for } \alpha < 4 \tag{6.18}$$

and

$$\int \frac{\chi_\mu(k)}{\omega(k)^\alpha}\|v^{\mathrm{ren}}\psi_\mu\|^2_{\mathcal{H}}\mathrm{d}k = \begin{cases} O(ln\mu) & if \quad \alpha = 4 \\ O(\mu^{4-\alpha}) & if \quad \alpha < 4, \end{cases} \tag{6.19}$$

where χ_μ is the infrared cutoff function.

Proof. (6.16) is obtained by direct computation, (6.17)is a consequence of Corollary 4.1. Then (6.18) and (6.19) follow easily by writing $v^{\mathrm{ren}}(k)\psi_\mu = v^{\mathrm{ren}}(k)\langle X\rangle^{-1}\langle X\rangle\psi_\mu$ and using (6.16) and (6.17). □

We need some uniform bounds on ψ_μ.

Lemma 6.5. *Assume (I) and (B). Then for μ small enough*

$$(\psi_\mu, N\psi_\mu) \leq C \ \textit{uniformly in } \mu > 0.$$

Proof. By the pullthrough formula

$$\begin{aligned} (\psi_\mu, N\psi_\mu) &\leq \int \|a(k)\psi_\mu\|^2_{\mathcal{H}}\mathrm{d}k = \int \|(H_\mu^{\mathrm{ren}} - E_\mu + \omega(k))^{-1} v_\mu^{\mathrm{ren}}\psi_\mu\|^2_{\mathcal{H}}\mathrm{d}k \\ &\leq \int \tfrac{1}{\omega(k)^2}\|v_\mu^{\mathrm{ren}}\psi_\mu\|^2_{\mathcal{H}}\mathrm{d}k \leq C \end{aligned}$$

uniformly in μ because of (6.18). □

Lemma 6.6. *Let $H_0 := K^{\mathrm{ren}} \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega)$. Then*

$$(\psi_\mu, H_0\psi_\mu) \leq C \ \textit{uniformly in } \mu > 0.$$

Proof. As quadratic form H_μ^{ren} is equivalent to H_0 uniformly in μ. □

Lemma 6.7. *Let $E := \inf \sigma(H^{\mathrm{ren}})$, $E_\mu := \inf \sigma(H^{\mathrm{ren}}_\mu)$. Assume (I) and (B). Then*

$$E - E_\mu = O(\mu).$$

Proof. Let $0 < \mu' < \mu$. We have

$$E_{\mu'} - E_\mu \le (\psi_\mu, (H^{\mathrm{ren}}_{\mu'} - H^{\mathrm{ren}}_\mu)\psi_\mu) = (\psi_\mu, (W_{\mu'} - W_\mu)\psi_\mu) + (\psi_\mu, \Phi(v^{\mathrm{ren}}_{\mu'} - v^{\mathrm{ren}}_\mu)\psi_\mu),$$
$$E_\mu - E_{\mu'} \le (\psi_{\mu'}, (H^{\mathrm{ren}}_\mu - H^{\mathrm{ren}}_{\mu'})\psi_{\mu'}) = (\psi_{\mu'}, (W_{\mu'} - W_\mu)\psi_{\mu'}) + (\psi_{\mu'}, \Phi(v^{\mathrm{ren}}_{\mu'} - v^{\mathrm{ren}}_\mu)\psi_{\mu'}).$$

Notice that $|W_{\mu'}(X) - W_\mu(X)| \le C(\mu' - \mu)$ uniformly in X, hence

$$(\psi_\mu, (W_{\mu'} - W_\mu)\psi_\mu) \le C|\mu' - \mu|.$$

Writing $\psi_\mu = \langle X\rangle^{-1}\langle X\rangle\psi_\mu$, using Schwarz inequality and $\|a(h)\psi\| \le \|h\|_{\mathfrak{h}}(\psi, (\mathrm{N}+1)\psi)^{1/2}$, we obtain

$$(\psi_\mu, \Phi(v^{\mathrm{ren}}_{\mu'} - v^{\mathrm{ren}}_\mu)\psi_\mu) \le C\left(\int \|(v^{\mathrm{ren}}_{\mu'}(k) - v^{\mathrm{ren}}_\mu(k))\langle X\rangle^{-1}\|^2_{\mathcal{B}(\mathcal{K})}\mathrm{d}k\right)^{1/2}(\psi_\mu, (\mathrm{N}+1)\psi_\mu)^{1/2}\|\langle X\rangle\psi_\mu\|_{\mathcal{H}}.$$

The last two terms of the right hand side product are bounded uniformly in μ by Lemmas 6.5 and (6.17). Hence by (6.16)

$$(\psi_\mu, \Phi(v^{\mathrm{ren}}_{\mu'} - v^{\mathrm{ren}}_\mu)\psi_\mu) \le C(\mu' - \mu)^2.$$

Estimating in the same way $E_\mu - E_{\mu'}$, we obtain $|E_\mu - E_{\mu'}| \le C|\mu' - \mu|$. Since $E = \lim_{\mu\to 0} E_\mu$ the lemma follows by letting $\mu' \to 0$. □

Proposition 6.1. *$a(k)\psi_\mu - (E - H^{\mathrm{ren}} - \omega(k))^{-1}v^{\mathrm{ren}}(k)\psi_\mu \to 0$ when $\mu \to 0$ in $L^2(\mathbb{R}^3, \mathrm{d}k; \mathcal{H})$.*

Proof. By the pullthrough formula

$$\begin{aligned}
& a(k)\psi_\mu - (E - H^{\mathrm{ren}} - \omega(k))^{-1}v^{\mathrm{ren}}(k)\psi_\mu \\
= \;& (E_\mu - H^{\mathrm{ren}}_\mu - \omega(k))^{-1}v^{\mathrm{ren}}_\mu(k)\psi_\mu - (E - H^{\mathrm{ren}} - \omega(k))^{-1}v^{\mathrm{ren}}(k)\psi_\mu \\
= \;& -(1-\chi_\mu)(k)(E - H^{\mathrm{ren}} - \omega(k))^{-1}v^{\mathrm{ren}}(k)\psi_\mu \\
& +(E_\mu - E)(E - H^{\mathrm{ren}} - \omega(k))^{-1}(E_\mu - H^{\mathrm{ren}}_\mu - \omega(k))^{-1}v^{\mathrm{ren}}_\mu(k)\psi_\mu \\
& +(E - H^{\mathrm{ren}} - \omega(k))^{-1}(W(X) - W_\mu(X))(E_\mu - H^{\mathrm{ren}}_\mu - \omega(k))^{-1}v^{\mathrm{ren}}_\mu(k)\psi_\mu \\
& +(E - H^{\mathrm{ren}} - \omega(k))^{-1}(\Phi(v^{\mathrm{ren}}) - \Phi(v^{\mathrm{ren}}_\mu))(E_\mu - H^{\mathrm{ren}}_\mu - \omega(k))^{-1}v^{\mathrm{ren}}_\mu(k)\psi_\mu \\
=: \;& R_{\mu,1}(k) + R_{\mu,2}(k) + R_{\mu,3}(k) + R_{\mu,4}(k).
\end{aligned}$$

Note that because of the ultraviolet cutoff, $v^{\mathrm{ren}}(k)$ is compactly supported in k. Therefore the behaviour of the terms for large k is not relevant. First we estimate $R_{\mu,1}(k)$:

$$\|R_{\mu,1}(k)\|_{\mathcal{H}} \le \mathbb{1}_{\{\omega(k)\le\mu\}}(k)\frac{1}{\omega(k)}\|v^{\mathrm{ren}}(k)\psi_\mu\|_{\mathcal{H}},$$

which by (6.18) implies $R_{\mu,1} \in o(\mu)$ in $L^2(\mathbb{R}^3, \mathrm{d}k; \mathcal{H})$.

Now we estimate $R_{\mu,2}(k)$. By Lemma 6.7, $E - E_\mu = O(\mu)$, then

$$\|R_{\mu,2}(k)\|_{\mathcal{H}} \le \tfrac{O(\mu)}{\omega(k)^2}\|v^{\mathrm{ren}}_\mu(k)\psi_\mu\|_{\mathcal{H}},$$

hence by (6.19) $\|R_{\mu,2}\|_{L^2(\mathbb{R}^3,\mathrm{d}k;\mathcal{H})} = O(\mu \ln^{1/2}\mu)$.
The same bound holds for $R_{\mu,3}$, noticing that $|W(X) - W_\mu(X)| \le O(\mu)$ uniformly in X.

Finally we estimate $R_{\mu,4}$. Let $\chi \in C_0^\infty(\mathbb{R})$ be a function such that $\operatorname{supp}\chi \subset]-\infty, \Sigma(H)[$. By Corollary

4.1 and Proposition 2.1 $\chi(H^{\rm ren})\langle X\rangle$ is a bounded operator. Since $E\notin \operatorname{supp}(1-\chi)$, the following estimate holds for all $u\in\mathcal{H}$, for $\lambda>0$:

$$\begin{aligned}\left\|(E-H-\lambda)^{-1}u\right\| &\leq \left\|(E-H-\lambda)^{-1}\chi(H)\langle X\rangle\langle X\rangle^{-1}u\right\| \\ &+\left\|(E-H-\lambda)^{-1}(1-\chi(H))u\right\| \leq \tfrac{C}{\lambda}\left\|\langle X\rangle^{-1}u\right\|+C\left\|u\right\|.\end{aligned}$$

Hence

$$\begin{aligned}\|R_{\mu,4}(k)\|_{\mathcal{H}} \leq \tfrac{C}{\omega(k)}&\|\Phi(\langle X\rangle^{-1}(v^{\rm ren}-v^{\rm ren}_\mu))(E_\mu-H^{\rm ren}_\mu-\omega(k))^{-1}v^{\rm ren}_\mu(k)\psi_\mu\|_{\mathcal{H}}\\ &+\|\Phi(v^{\rm ren}-v^{\rm ren}_\mu)(E_\mu-H^{\rm ren}_\mu-\omega(k))^{-1}v^{\rm ren}_\mu(k)\psi_\mu\|_{\mathcal{H}}.\end{aligned}$$

Since $\left\|\phi(v)(H_0+C)^{-1/2}\right\|\leq\left(\int\|v(k)\|^2_{\mathcal{B}(\mathcal{K})}\left(\frac{1}{\omega(k)}+1\right)\mathrm{d}k\right)^{1/2}$ and $\left\|(H_0+C)^{-1/2}(E_\mu-H_\mu-\omega(k))^{-1}\right\|\leq\frac{C}{\omega(k)}$ one obtains

$$\begin{aligned}\|R_{\mu,4}(k)\|_{\mathcal{H}}\leq \tfrac{C}{\omega^2(k)}&\left(\int\|\langle X\rangle^{-1}(v^{\rm ren}-v^{\rm ren}_\mu)(k)\|^2_{\mathcal{B}(\mathcal{K})}\left(\tfrac{1}{\omega(k)}+1\right)\mathrm{d}k\right)^{1/2}\|v^{\rm ren}_\mu(k)\psi_\mu\|_{\mathcal{H}}\\ &+\tfrac{C}{\omega(k)}\left(\int\|(v^{\rm ren}-v^{\rm ren}_\mu)(k)\|^2_{\mathcal{B}(\mathcal{K})}\left(\tfrac{1}{\omega(k)}+1\right)\mathrm{d}k\right)^{1/2}\|v^{\rm ren}_\mu(k)\psi_\mu\|_{\mathcal{H}}.\end{aligned}$$

By writing $v^{\rm ren}-v^{\rm ren}_\mu=v^{\rm ren}(1-\chi_\mu)$ and by (6.16), one can easily check that:

$$\left(\int\|\langle X\rangle^{-1}(v^{\rm ren}-v^{\rm ren}_\mu)(k)\|^2_{\mathcal{B}(\mathcal{K})}\left(\frac{1}{\omega(k)}+1\right)\mathrm{d}k\right)^{1/2}=O(\mu^{3/2}),$$

and

$$\left(\int\|(v^{\rm ren}-v^{\rm ren}_\mu)(k)\|^2_{\mathcal{B}(\mathcal{K})}\left(\frac{1}{\omega(k)}+1\right)\mathrm{d}k\right)^{1/2}=O(\mu^{1/2}).$$

Then by (6.19) $\|R_{\mu,4}\|_{L^2(\mathbb{R}^3,\mathrm{d}k;\mathcal{H})}=o(\mu)$. □

Lemma 6.8. *Let us denote $T(k):=(E-H^{\rm ren}-\omega(k))^{-1}v^{\rm ren}(k)\langle X\rangle^{-1}$. Then*

$$T(k) \text{ belongs to } L^2(\mathbb{R}^3,\mathrm{d}k;B(\mathcal{H})), \tag{6.20}$$

and

$$\|T(k)-T(k+s)\|_{L^2(\mathbb{R}^3,\mathrm{d}k;B(\mathcal{H}))}\to 0 \quad \text{as} \quad s\to 0. \tag{6.21}$$

Remark 6.1. Note that in general (6.21) does *not* follow from (6.20) since $B(\mathcal{H})$ is *not* a separable Banach space, but is verified for the specific element $T(k)$.

Proof. Set $\mathfrak{H}:=L^2(\mathbb{R}^3,\mathrm{d}k;B(\mathcal{H}))$. We have $\|T(k)\|^2_{B(\mathcal{H})}\leq\frac{1}{\omega(k)^2}\|v_\mu(k)\langle X\rangle^{-1}\|^2_{\mathcal{B}(\mathcal{K})}$ which is integrable by (6.16). This prove (6.20).
For $0<C_1<C_2$, let us denote $K_2:=[0,C_1[$, $K_2:=[C_1,C_2[$ and $G:=[C_2,\infty[$; then we can write $\mathbb{1}=\mathbb{1}_{K_1}(|k|)+\mathbb{1}_{K_2}(|k|)+\mathbb{1}_G(|k|)$, so $T(k)=\mathbb{1}_{K_1}(|k|)T(k)+\mathbb{1}_{K_2}(|k|)T(k)+\mathbb{1}_G(|k|)T(k)$. So we can write

$$\begin{aligned}T(k+s)-T(k)&=\mathbb{1}_{K_1}(|k+s|)T(k+s)-\mathbb{1}_{K_1}(|k|)T(k)\\ &+\mathbb{1}_{K_2}(|k+s|)T(k+s)-\mathbb{1}_{K_2}(|k|)T(k)+\mathbb{1}_G(|k+s|)T(k+s)-\mathbb{1}_G(|k|)T(k).\end{aligned}$$

We have

$$\begin{aligned}&\|\mathbb{1}_{K_1}(|k+s|)T(k+s)-\mathbb{1}_{K_1}(|k|)T(k)\|_{\mathfrak{H}}\leq 2\left\|\mathbb{1}_{K_1}(|k|)T(k)\right\|_{\mathfrak{H}}\\ &\|\mathbb{1}_{G}(|k+s|)T(k+s)-\mathbb{1}_{G}(|k|)T(k)\|_{\mathfrak{H}}\leq 2\left\|\mathbb{1}_{G}(|k|)T(k)\right\|_{\mathfrak{H}},\end{aligned}$$

but on the other hand $\left\|\mathbb{1}_{K_1}(|k|)T(k)\right\|_{\mathfrak{H}}\to 0$ as $C_1\to 0$ and $\left\|\mathbb{1}_{G}(|k|)T(k)\right\|_{\mathfrak{H}}\to 0$ as $C_2\to\infty$, since $T(k)\in\mathfrak{H}$. Let us now fix C_1 and C_2. We can write

$$\begin{aligned}\mathbb{1}_{K_2}(|k+s|)T(k+s)-\mathbb{1}_{K_2}(|k|)T(k)= \quad &(\mathbb{1}_{K_2}(|k+s|)-\mathbb{1}_{K_2}(|k|))T(k+s)\\ -\mathbb{1}_{K_2}(|k|)(T(k+s)-T(k))=:T_1+T_2.&\end{aligned}$$

By dominated convergence $\|T_1\|_{\mathfrak{H}}^2 \to 0$ as $s \to 0$. Now

$$\|T_2\|_{\mathfrak{H}}^2 \quad = \int \mathbb{1}_{K_2}(|k|)\, \|T(k+s)-T(k)\|_{B(\mathcal{H})}^2\, \mathrm{d}k \le \int_{C_1/2}^{2C2} \|T(k+s)-T(k)\|_{B(\mathcal{H})}^2\, \mathrm{d}k$$

for $s < C_1/4$. Next we have

$$\begin{aligned} &T(k+s)-T(k) = (E-H^{\mathrm{ren}}-\omega(k))^{-1}(v^{\mathrm{ren}}(k+s)-v^{\mathrm{ren}}(k))\langle X\rangle^{-1} \\ &\quad +(E-H^{\mathrm{ren}}-\omega(k))^{-1}(E-H^{\mathrm{ren}}-\omega(k+s))^{-1}v^{\mathrm{ren}}(k+s)(\omega(k+s)-\omega(k))\langle X\rangle^{-1}, \end{aligned}$$

so

$$\begin{aligned} &\|T(k+s)-T(k)\|_{B(\mathcal{H})} \le \tfrac{1}{\omega(k)}\|(v^{\mathrm{ren}}(k+s)-v^{\mathrm{ren}}(k))\langle X\rangle^{-1}\|_{\mathcal{B}(\mathcal{K})} \\ &\quad +\tfrac{1}{\omega(k)}\tfrac{1}{\omega(k+s)}\|(v^{\mathrm{ren}}(k+s)\langle X\rangle^{-1}\|_{\mathcal{B}(\mathcal{K})}(|k+s|-|k|) \\ &\quad \le C(C_1,C_2)\|(v^{\mathrm{ren}}(k+s)-v^{\mathrm{ren}}(k))\langle X\rangle^{-1}\|_{\mathcal{B}(\mathcal{K})} + C(C_1,C_2)|s|\|v^{\mathrm{ren}}(k)\langle X\rangle^{-1}\|_{\mathcal{B}(\mathcal{K})}^2 \end{aligned}$$

uniformly for $C_1/2 < |k| < 2C_2$ and $|s| < C_1/4$ where $C(C_1,C_2)$ is a constant which depends on C_1 and C_2. Since, as one can easily verify, for arbitrary $0 < D_1 < D_2$

$$\lim_{s\to 0}\int_{D_1<|k|<D_2} \left\|(v^{\mathrm{ren}}(k)-v^{\mathrm{ren}}(k+s))\langle X\rangle^{-1}\right\|^2 \mathrm{d}k = 0,$$

we can conclude

$$\lim_{s\to 0}\int_{C_1/2}^{2C2} \|T(k+s)-T(k)\|_{B(\mathcal{H})}^2\, \mathrm{d}k = 0.$$

By fixing first $C_1 << 1$ and $C_2 >> 1$, letting then $s \to 0$, the proof is concluded. □

We recall the following:

Proposition 6.2. *Let $f \in L^2(\mathbb{R}^d, \mathrm{d}k; \mathcal{B})$ where $\mathcal{B}$ is a Banach space and let us denote U_s the group of isometries given by $U_s f(k) := f(k+s)$. Suppose $\|f - U_s f\| \to 0$ as $s \to 0$. Then, for any $F \in C_0^\infty(\mathbb{R}^d)$ such that $F(0) = 1$,*

$$\left\|1 - F(\frac{D_k}{R})f\right\| \to 0 \quad as \quad R \to \infty$$

where $F(\frac{D_k}{R})f = (2\pi)^{-d}\int \hat{F}(s)U_{-R^{-1}s}f ds$.

Lemma 6.9. *Let $F \in C_0^\infty(\mathbb{R})$ be a cutoff function with $0 \le F \le 1$, $F(s) = 1$ for $|s| \le 1/2$, $F(s) = 0$ for $|s| \ge 1$. Let $F_R(x) = F(\frac{|x|}{R})$. Then*

$$\lim_{\mu\to 0,\, R\to+\infty}(\psi_\mu, \mathrm{d}\Gamma(1-F_R)\psi_\mu) = 0.$$

Proof. Set $\mathfrak{H} := L^2(\mathbb{R}^3, \mathrm{d}k; B(\mathcal{H}))$. As in [12, Lemma 4.5], we obtain

$$(\psi_\mu, \mathrm{d}\Gamma(1-F_R)\psi_\mu) \le \|T(k)\|_{\mathfrak{H}}\|(1-F(\tfrac{|D_k|}{R}))T(k)\|_{\mathfrak{H}}\|\langle X\rangle\psi_\mu\|_{\mathcal{H}} + o(\mu^0).$$

By Lemma 6.8, $\|T(k)-T(k+s)\|_{\mathfrak{H}} \to 0$ as $s \to 0$, hence by Proposition 6.2

$$\|(1-F(\frac{|D_k|}{R}))T(k)\|_{\mathfrak{H}} \in o(R^0).$$

So we can conclude that $(\psi_\mu, \mathrm{d}\Gamma(1-F_R)\psi_\mu) = o(R^0) + o(\mu^0)$. □

Proof of Theorem 1.2: as in [12, Theorem 1], by replacing the compact operator $\chi(N \le \lambda)\chi(H_0 \le \lambda)\Gamma(F_R)$ (where χ is a smoothed characteristic function of the unit ball) by the compact operator $\chi(N \le \lambda)\chi(H_0 \le \lambda)\Gamma(F_R)\chi_P(|X|)$, and using in addition that, as a consequence of Corollary 4.1, for any $\delta > 0$ we can choose P large enough such that $\|(1-\chi_P)(|X|)\psi_\mu\| \le \delta$, uniformly in μ for $\mu < \mu_0$. □

A Appendix A

In this section we give a sketch of the proof of Lemma 4.1. The idea behind the proof is to compare $\hat{H}_\mu$ with an auxiliary Hamiltonian where the electrons are localized in some regions, and the photons are localized near the electrons.

We recall the following fact about existence of some partitions of unity.

Proposition A.1. *There exists a family of functions $F_a : \mathbb{R}^{3N} \to \mathbb{R}$, for $a \subset \{1, ..., N\}$ such that*

(i) $\sum_a F_a^2 = 1$,

(ii) for all $a \neq \emptyset$ $\operatorname{supp} F_a \subset \{ X \in \mathbb{R}^{3N} \mid |X| \geq 1, \min_{i \in a^c\, j \in a}(|\mathrm{x}_i - \mathrm{x}_j|, |\mathrm{x}_j|) \geq C\}$ *where C is some positive constant,*

(iii) if $a = \emptyset$, then $\operatorname{supp} F_a$ *is compact,*

(iv) let $\sharp a$ be the cardinality of the set a; the functions $\sum_{\sharp a = p} F_a^2$ are symmetric for all $0 \leq p \leq N$.

Proof. see for example [8]. □

With this notation the subset a will represent the particles far from the origin.
Each function of the family will be used to localize fermions. Corresponding to each fermion localization we now define boson localization. For a given a, consider the function

$$\begin{aligned} g_{\infty,\,a,\,P}(x, X) &:= \begin{cases} \prod_{j \in a} 1 - \chi(\frac{x - \mathrm{x}_j}{P}) & a \neq \emptyset \\ \chi(\frac{x}{P}) & a = \emptyset \end{cases} \\ g_{0,\,a,\,P}(x, X) &:= \quad 1 - g_{\infty,\,a}(x, X) \end{aligned}$$

where χ is a smoothed characteristic function of the unit ball.
Now let us set for $\varepsilon = 0, \infty$:

$$j_{\varepsilon,\,a,\,P} := j_{\varepsilon,a}(x, X) := \frac{g_{\varepsilon,\,a,\,P}(x, X)}{\sqrt{g_{\infty,\,a,\,P}(x, X)^2 + g_{0,a,\,P}(x, X)^2}}$$

so that $j_{0,\,a,\,P}^2 + j_{\infty,\,a,\,P}^2 = 1$.

Remark A.1. Note that for $a \neq \emptyset$

$$\begin{aligned} \operatorname{supp} j_{\infty,\,a,\,P} &\subset \{x \in \mathbb{R}^3 \mid |x - \mathrm{x}_j| > P, \text{ for all } j \in a\} \\ \operatorname{supp} j_{0,\,a,\,P} &\subset \{x \in \mathbb{R}^3 \mid |x - \mathrm{x}_j| \leq P, \text{ for some } j \in a\} \end{aligned}.$$

For each a we define

$$K_a := K - (\textstyle\sum_{i \in a, j \notin a} w(\mathrm{x}_i - \mathrm{x}_j) + \sum_{i \in a} v(\mathrm{x}_j)). \tag{A.22}$$

Now we define the cluster Hamiltonian $\hat{H}_{a,\,\mu}$ by

$$\hat{H}_{a,\,\mu} := K_a \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega_\mu) + \lambda \Phi(v_\mu). \tag{A.23}$$

The next lemma follow easily from hypothesis *(I)*.

Lemma A.1. *Let $\hat{H}_\mu$ be the Hamiltonian defined in (2.8). Assume (I). Then $F_{a,R}(\hat{H}_\mu - \hat{H}_{a,\,\mu}) = O(R^{-\varepsilon})$ for all a, where $\varepsilon := inf\{\varepsilon_1, \varepsilon_2\}$.*

In order to deal with photon localization, we need to introduce the *extended cluster Hamiltonians* $\hat{H}^{\text{ext}}_{a,\,\mu}$. We introduce the space $\mathcal{H}^{\text{ext}} := \mathcal{K} \otimes \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h})$, on which we define the following operators:

$$\mathrm{d}\Gamma^{\text{ext}}(\omega_\mu) := \mathbb{1}_{\mathcal{K}} \otimes \mathrm{d}\Gamma(\omega_\mu) \otimes \mathbb{1}_{\Gamma(\mathfrak{h})} + \mathbb{1}_{\mathcal{K}} \otimes \mathbb{1}_{\Gamma(\mathfrak{h})} \otimes \mathrm{d}\Gamma(\omega_\mu),$$

$$\Phi^{\text{ext}}_a(v_\mu) := \sum_{j \notin a} \Phi(v_{\mu,j}) \otimes \mathbb{1}_{\Gamma(\mathfrak{h})} + \mathbb{1}_{\Gamma(\mathfrak{h})} \hat{\otimes} \sum_{j \in a} \Phi(v_{\mu,j}),$$

where $v_{\mu,j} := v_\mu(\mathrm{x}_j, k)$.
We define

$$\hat{H}^{\text{ext}}_{a,\,\mu} := K_a \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma^{\text{ext}}(\omega_\mu) + \lambda \Phi^{\text{ext}}_a(v_\mu)$$

The extended cluster Hamiltonians are built *ad hoc* in order to have, up to identifications,

$$\hat{H}^{\text{ext}}_{a,\,\mu} \quad = \quad H_{N',\,\mu} \otimes \mathbb{1}_a \otimes \mathbb{1}_{\Gamma(\mathfrak{h})} + \mathbb{1}_{a^C} \otimes \mathbb{1}_{\Gamma(\mathfrak{h})} \otimes H_{N-N',\,\mu}$$

where $\sharp a = N - N'$. This implies $\inf \sigma(\hat{H}^{\text{ext}}_{a,\,\mu}) = E_{N',\,\mu} + E^0_{N-N',\,\mu}$.

The following Lemma is well known.

Lemma A.2. *Let $\hat{H}_\mu$ be the Hamiltonian defined in (2.10). Then there exist some constants $C, D \in \mathbb{R}^+$ such that*

$$\mathrm{N} \leq \frac{C\hat{H}_\mu + D}{\mu}.$$

Lemma A.3. *Let $\hat{H}_{a,\,\mu}$ the cluster Hamiltonian defined in (A.23). Let $j_{a,\,P} := (j_{\infty,\,a,\,P}, j_{0,\,a,\,P})$. Then*

(i) if $a \neq \emptyset$,

$$F_{a,R}\,(\hat{H}_{a,\,\mu} - \check{\Gamma}^*(j_{a,\,P})\hat{H}^{\text{ext}}_{a,\,\mu}\check{\Gamma}(j_{a,\,P})) = O(\frac{\ln^{1/2}\mu}{\mu P})(\hat{H}_\mu + C)$$

when $R = \gamma P$ with $\gamma >> 1$,

(ii) if $a = \emptyset$, the same holds when $P = \gamma R$ with $\gamma >> 1$.

Proof. *(i)* We have to evaluate

$$\begin{aligned}
&F_{a,R}\left(\hat{H}_{a,\,\mu} - \check{\Gamma}^*(j_{a,\,P})\hat{H}^{\text{ext}}_{a,\,\mu}\check{\Gamma}(j_{a,\,P})\right) = F_{a,R}\,\left(K_a \otimes \mathbb{1}_{\Gamma(\mathfrak{h})} + \check{\Gamma}^*(j_{a,\,P})(K_a \otimes \mathbb{1}_{\Gamma(\mathfrak{h})} \otimes \mathbb{1}_{\Gamma(\mathfrak{h})})\check{\Gamma}(j_{a,\,P})\right)\\
&\quad + F_{a,R}\,\left(\mathrm{d}\Gamma(\omega_\mu) - \check{\Gamma}^*(j_{a,\,P})\mathrm{d}\Gamma^{\text{ext}}(\omega_\mu)\check{\Gamma}(j_{a,\,P})\right) + \lambda F_{a,R}\,\left(\Phi(v_\mu) - \check{\Gamma}^*(j_{a,\,P})\Phi^{\text{ext}}_a(v_\mu)\check{\Gamma}(j_{a,\,P})\right)\\
&\quad =: I_1 + I_2 + I_3.
\end{aligned}$$

Note that $j_{a,\,P}$ is a function of *both* X and x.

Consider first I_1. Using (2.7) we have:

$$\begin{aligned}
I_1 &= F_{a,R}\left(\textstyle\sum_{i=1}^N \mathrm{d}\Gamma(ad^2_{j_{0,\,a,\,P}}\tfrac{1}{2}\triangle_{\mathrm{x}_i} + ad^2_{j_{\infty,\,a,\,P}}\tfrac{1}{2}\triangle_{\mathrm{x}_i})\right)\\
&\leq \textstyle\sum_{i=1}^N \left\| ad^2_{j_{0,\,a,\,P}}\tfrac{1}{2}\triangle_{\mathrm{x}_i} + ad^2_{j_{\infty,\,a,\,P}}\tfrac{1}{2}\triangle_{\mathrm{x}_i}\right\|_{\mathcal{B}(\mathcal{K}\otimes\mathfrak{h})}(\mathrm{N}+1) \leq O((\mu P)^{-1})(\hat{H}_\mu + C)
\end{aligned}$$

by Lemma A.2.

Consider now I_2. Using (2.7), we obtain

$$I_2 \leq \left\| ad^2_{j_{0,\,a,\,P}}\omega_\mu + ad^2_{j_{\infty,\,a,\,P}}\right\|_{\mathcal{B}(\mathfrak{h})}(\mathrm{N}+1) \leq O((\mu P)^{-1})(\hat{H}_\mu + C)$$

by Lemma A.2.

Consider now I_3. Using (2.6), it is easy to compute

$$\begin{aligned} I_3 &= \lambda F_{a,R}\,\Big(\check{\Gamma}^*(j_{a,\,P})\Big(\textstyle\sum_{j\notin a}\Phi((j_{\infty,\,a,\,P}-1)v_{\mu,j})\otimes \mathbb{1}_{\Gamma(\mathfrak{h})}+\mathbb{1}_{\Gamma(\mathfrak{h})}\hat{\otimes}\Phi(j_{0,\,a,\,P}v_{\mu,j}) \\ &\quad +\textstyle\sum_{j\in a}\Phi(j_{\infty,\,a,\,P}v_{\mu,j})\otimes \mathbb{1}_{\Gamma(\mathfrak{h})}+\mathbb{1}_{\Gamma(\mathfrak{h})}\otimes\Phi((j_{0,\,a,\,P}-1)v_{\mu,j})\Big)\,\check{\Gamma}(j_{a,\,P})\Big)\,. \end{aligned}$$

A term of the form $\tilde{\jmath}\,v_{\mu,j}$ (where $\tilde{\jmath}$ will be $j_{0,\,a,\,P}$, $j_{\infty,\,a,\,P}-1$, etc) can be seen in two ways: as an element of $\mathfrak{h}:=L^2(\mathbb{R}^3,\mathrm{d}k)$, in this case $\tilde{\jmath}$ is a pseudodifferential operator on $\mathfrak{h}$, in other words $\tilde{\jmath}\,v_{\mu,j}=\tilde{\jmath}\,(D_k)v_\mu(\mathrm{x}_j,k)$; or as an element of $\mathfrak{h}_x:=L^2(\mathbb{R}^3,\mathrm{d}x)$, in this case we mean $\tilde{\jmath}\,v_{\mu,j}=\tilde{\jmath}\,(x)v_\mu(x-\mathrm{x}_j)$ with $v_\mu(x-\mathrm{x}_j):=\mathcal{F}v_\mu(\mathrm{x}_j,k)$ where $\mathcal{F}$ is the Fourier transform with respect to the variable k. Anyway, by unitary of $\mathcal{F}$, $\|\tilde{\jmath}\,v_{\mu,j}\|_{\mathfrak{h}}=\|\tilde{\jmath}\,v_{\mu,j}\|_{\mathfrak{h}_x}$ (see also the proof of Lemma 4.2), so we can write $\|\tilde{\jmath}\,v_{\mu,j}\|$ without ambiguity.

Let's consider the terms of the form $A:=F_{a,R}\,\check{\Gamma}^*(j_{a,\,P})a^\sharp\,(\tilde{\jmath}\,v_{\mu,j})\otimes\mathbb{1}_{\Gamma(\mathfrak{h})}\check{\Gamma}(j_{a,\,P})$ (here λ is neglected). For $u\in\mathcal{H}$

$$\begin{aligned} &\|F_{a,R}\,\check{\Gamma}^*(j_{a,\,P})a^\sharp(\tilde{\jmath}\,v_{\mu,j})\otimes\mathbb{1}_{\Gamma(\mathfrak{h})}\check{\Gamma}(j_{a,\,P})u\|^2_{\mathcal{H}} \\ &\le \textstyle\int_{\mathrm{supp}\,F_{a,R}}\left\|(a^\sharp(\tilde{\jmath}\,v_{\mu,j})\otimes\mathbb{1}_{\Gamma(\mathfrak{h})})(\mathrm{N}^{\mathrm{ext}}+1)^{-1/2}\check{\Gamma}(j_{a,\,P})(\mathrm{N}+1)^{1/2}u(X)\right\|^2_{\Gamma(\mathfrak{h})\otimes\Gamma(\mathfrak{h})}\mathrm{d}X \\ &\le \textstyle\int_{\mathrm{supp}\,F_{a,R}}\|\tilde{\jmath}\,v_{\mu,j}\|^2_{\mathfrak{h}}\langle u(X),(\mathrm{N}+1)u(X)\rangle_{\Gamma(\mathfrak{h})}\mathrm{d}X \end{aligned}$$

since $\left\|a^\sharp(h)u\right\|^2\le\|h\|^2_{\mathfrak{h}}(u,(\mathrm{N}+1)\,u\,)$.

The same estimate holds for the terms of the form $A':=F_{a,R}\,\check{\Gamma}^*(j_{a,\,P})\mathbb{1}_{\Gamma(\mathfrak{h})}\otimes a^\sharp\,(\tilde{\jmath}\,v_j)\,\check{\Gamma}(j_{a,\,P})$.

Hence we have to estimate the norms:

$$\begin{array}{lll} \|j_{\infty,\,a,\,P}v_{\mu,j}\|\,, & \|(j_{0,\,a,\,P}-1)v_{\mu,j}\| & \text{for all } j\in a,\ \text{for } X\in\mathrm{supp}\,F_{a,R} \\ \|(j_{\infty,\,a,\,P}-1)v_{\mu,j}\|\,, & \|j_{0,\,a,\,P}v_{\mu,j}\| & \text{for all } j\notin a,\ \text{for } X\in\mathrm{supp}\,F_{a,R} \end{array}$$

Since $\mathrm{supp}\,j_{\infty,\,a,\,P}\subset\{x\in\mathbb{R}^3\mid|x-\mathrm{x}_j|>P,\ \text{for all } j\in a\}$, then by Lemma 4.2$\|j_{\infty,\,a,\,P}v_{\mu,j}\|=O(\ln^{1/2}\mu\;P^{-1})$ uniformly in x_j. The same holds for $\|(j_{0,\,a,\,P}-1)v_{\mu,j}\|$.

Now $\mathrm{supp}\,j_{0,\,a,\,P}\subset\{x\in\mathbb{R}^3\mid|x-\mathrm{x}_i|\le P,\ \text{for some } i\in a\}$ but on the other hand $X\in\mathrm{supp}\,F_{a,R}$ implies $|\mathrm{x}_j-\mathrm{x}_i|>R$. Choosing $R=\gamma P$ with $\gamma>>1$, we obtain, for $j\notin a$ and $X\in\mathrm{supp}\,F_{a,R}$, $|x-\mathrm{x}_j|>(\gamma-1)P$, $\mathrm{supp}\,j_{0,\,a,\,P}\subset\{x\in\mathbb{R}^3\mid|x-\mathrm{x}_j|\le(\gamma-1)P\}$. Hence by Lemma 4.2 $\|j_{0,\,a,\,P}v_{\mu,j}\|^2=O(\ln\mu^{1/2}\;P^{-1})$ uniformly in x_j; the same holds for $\|(j_{\infty,\,a,\,P}-1)v_{\mu,j}\|$. Collecting the estimates for I_1, I_2, I_3 we obtain the lemma.

(ii) We proceed in the same way. Since $a=\emptyset$, we only have to evaluate norms of the type $\|j_{0,\,a,\,P}v_{\mu,j}\|$. In this case $j_{\infty,\,a,\,P}$ is compactly supported and $\mathrm{supp}\,j_{0,\,a,\,P}\subset\{x\in\mathbb{R}^3\mid|x|>P\}$, but also $F_{a,R}$ is compactly supported *i.e.* $|X|<R$. Hence, in this case we have to choose $R<<P$, for example $P=\gamma R$ with $\gamma>>1$ so that $|x-\mathrm{x}_j|>\frac{(\gamma-1)}{\gamma}P$, hence by Lemma 4.2 $\|j_{0,\,a,\,P}v_{\mu,j}\|=O(\ln^{1/2}\,\mu\;P^{-1})$. □

Corollary [Lemma 4.1] *Let $\hat{H}_\mu$ be the Hamiltonian defined in (2.8). Then*

$$\hat{H}_\mu\ge\tau_\mu-f(\mu)o\left(R^0\right)(\hat{H}_\mu+C)\ \text{ on } D_R,$$

where $f(\mu):=\frac{\ln^{1/2}\mu}{\mu}$ and $D_R:=\{\psi\in\mathcal{H}\,|\,\psi(X)=0 \text{ if } |X|<R\}$.

Proof. Easy using the previous Lemma and IMS localization formula. See [13]. □

B Appendix B

Theorem B.1 (functional calculus formula)**.** *Let $f \in C_0^\infty(\mathbb{R})$ and H be a self-adjoint operator on a Hilbert space, then there exists a function $\tilde{f} \in C_0^\infty(\mathbb{C})$ such that $\tilde{f}_{|\mathbb{R}} = f$, $|\frac{\partial \tilde{f}}{\partial \bar{z}}| \leq c_n |\mathrm{Im} z|^n$ for all $n \in \mathbb{N}$ and*

$$f(H) = \frac{1}{\mathrm{i}\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}}(z)(z-H)^{-1} dz \wedge d\overline{z}.$$

The function $\tilde{f}$ is called an *almost-analytic extension* of f .

Proof. See for instance [15]. □

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